Ill-posedness of the two-dimensional Keller–Segel model in Triebel-Lizorkin spaces

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Abstract

This article proves the ill-posedness of the Cauchy problem for the two-dimensional Keller–Segel model in Triebel–Lizorkin spaces, $\dot{F}_2^{-1,r}(\mathbb{R}^2)$ for $2 < r \leq \infty$. In particular, it is shown that solutions can develop norm inflation under certain settings in that the solution can become arbitrarily large after an arbitrarily short time even for small initial data.

Keywords: Keller–Segel model, Triebel–Lizorkin space, Ill-posedness.
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1. Introduction

In this article, we study the ill-posedness of a well-known chemotaxis model in two dimensions, the Keller–Segel model of the parabolic–parabolic type,

$$\partial_t u - \Delta u + \nabla \cdot (u \nabla v) = 0 \qquad \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \tag{1.1}$$

$$\partial_t v - \Delta v - u = 0 \qquad \text{in } \mathbb{R}_+ \times \mathbb{R}^2, \tag{1.2}$$

$$(u, v)|_{t=0} = (u_0, v_0) \qquad \text{in } \mathbb{R}^2. \tag{1.3}$$

Here, $\mathbb{R}_+ := (0, \infty)$, $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^2$, $u = u(t, x)$ and $v = v(t, x)$ are the scalar-valued density of amoebae and the scalar-valued concentration of chemical attractant, respectively, while (u_0, v_0) are the given initial data. The term chemotaxis refers to the attraction and movement of cellular organisms such as amoebae or bacteria in response to chemical stimulation. The Keller–Segel model, first introduced by Keller and Segel in [17], is perhaps the most common model for describing this motion of cell migration through chemical attraction. For more details on the model and its physical derivation, we refer the reader to [17] and the work of Childress and Percus in [8].

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When studying such nonlinear physical systems, there are several primary aspects of concern. One aspect is on the basic property of local or global-in-time well-posedness of the problem. We may ask if solutions exist in some sense, are they unique, and do they vary continuously upon small perturbations of the initial data. A closely related and important aspect is on the finite-time blowup of the solutions. Another related aspect concerns the setting in which the model is ill-posed. In fact, the main objective of this article concerns the thorough analytical examination of this model by identifying the proper functional space setting in terms of the Triebel–Lizorkin spaces in which the Cauchy problem is ill-posed. More specifically, we examine the critical or dividing number with respect to r for the well-posedness of solutions in the homogeneous Triebel–Lizorkin space, $\dot{F}_2^{-1,r}(\mathbb{R}^2)$. Remarkably, for the two-dimensional Keller–Segel model, our main result suggests that the critical number is $r = 2$. Here, by the critical number we mean that well-posedness holds for $r = 2$, but the system is, in fact, ill-posed in $\dot{F}_2^{-1,r}(\mathbb{R}^2)$ for $2 < r \leq \infty$. Let us be more precise in our description of the results in this paper. When we refer to the well-posedness (or ill-posedness) for (1.1)–(1.3) in Triebel–Lizorkin spaces, we mean the well-posedness (or ill-posedness) of mild solutions for initial data $(u_0, v_0) \in \dot{F}_2^{-1,r}(\mathbb{R}^2) \times \dot{F}_\infty^{0,2}(\mathbb{R}^2)$. As a result of establishing this dichotomy between well-posedness and ill-posedness, we find the critical setting in which the model remains valid while gaining a deeper understanding of the setting in which the model fails to capture even the most basic deterministic features.

To show the ill-posedness of system (1.1)–(1.3), we implement the novel framework of norm-inflation pioneered by Bourgain and Pavlović [5] in their study of the ill-posedness of the Navier–Stokes equation in the largest critical space $\dot{B}_\infty^{-1,\infty}$; but in doing so, we contribute new approaches and ideas by adopting this technique in our examination of the Keller–Segel model.

1.1. Remarks on the Well-posedness and Finite-time Blow-up

We mention that the set of equations (1.1)–(1.2) is scale invariant since both equations,

$$\partial_t u - \Delta u + \nabla \cdot (u \nabla v) = 0 \text{ and } \partial_t v - \Delta v - u = 0,$$

are scale invariant under the transformations

$$(u(t,x), v(t,x)) \to (\lambda^2 u(\lambda^2 t, \lambda x), v(\lambda^2 t, \lambda x)) \quad \text{for all } \lambda > 0.$$

The idea of using a functional setting invariant by scaling is now classical and originates from several works. For instance, for more on the global existence of mild solutions to system (1.1)–(1.3) with initial data $(u_0, v_0) \in H^{\frac{n}{r}-2,r}(\mathbb{R}^n) \times H^{\frac{n}{r},r}(\mathbb{R}^n)$ with $\max\{1, \frac{n}{4}\} < r < \frac{n}{2}$, see [19]; for initial data $(u_0, v_0) \in L_w^{n/2}(\mathbb{R}^n) \times BMO(\mathbb{R}^n)$ with $n \geq 3$, see [18]; and for initial data $(u_0, v_0) \in L^{\frac{n}{2}}(\mathbb{R}^n) \times \dot{H}^{2\alpha, \frac{n}{2\alpha}}(\mathbb{R}^n)$ with $n \geq 3$ and $\frac{n}{2(n+2)} < \alpha \leq \frac{1}{2}$, see [20]. In [9], Deng and Li proved the global-in-time existence and uniqueness for the Cauchy problem (1.1)–(1.3) with initial data in $L^1(\mathbb{R}^2) \times L^\infty(\mathbb{R}^2)$ and proved the existence and uniqueness of mild solutions for initial data in $H_b^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$. In addition to results on the existence and uniqueness of mild solutions in scale invariant spaces, studies on the asymptotic behavior of solutions can be found in [16, 26], and studies on stationary solutions can be found in [13, 22]. The reader is referred to [15] and the references therein for results concerning the quasilinear degenerate Keller–Segel system. In addition, the finite-time blowup of solutions has been studied for the simpler Keller–Segel model of the parabolic-elliptic type (i.e. $v_t = 0$ in (1.2)) in [2, 3, 4, 14, 25]. For this system, it is known that there is a critical threshold number for the initial density such that global-in-time well-posedness

holds for values below this threshold number and finite-time blowup occurs for values above this threshold number. For the parabolic-parabolic type, analogous results on finite-time blowup have remained relatively open, however, the critical mass threshold and global well-posedness have been studied in [6, 24].

1.2. Basic Notions of Norm Inflation

Let us describe the general idea for showing ill-posedness via norm inflation, but first, let us recall the definition of well-posedness. A Cauchy problem is said to be *locally well-posed* in Z if for every initial data $u_0 \in Z$ there exists a time $T = T(u_0) > 0$ such that

(1) a solution of the initial value problem exists in the time interval $[0, T]$,

(2) is unique in a certain Banach space of functions $Y \subset C([0,T]; Z)$ or $Y \subset C_w([0,T]; Z)$,

(3) the solution map from initial u_0 to solution u is continuous from Z to $C([0,T); Z)$ or $C_w([0,T]; Z)$.

Furthermore, if T can be taken arbitrarily large, we say that the Cauchy problem is *globally well-posed*, and we say the Cauchy problem is *ill-posed* if it is not well-posed. By solutions to the Keller–Segel model, we mean mild solutions to the equivalent system of integral equations as follows:

$$u = e^{t\Delta} u_0 - B(u, v), \tag{1.4}$$

$$v = e^{t\Delta} v_0 + L(u), \tag{1.5}$$

where

$$B(u, v) := \int_0^t e^{(t-\tau)\Delta} \nabla \cdot (u \nabla v) \, d\tau \quad \text{and} \quad L(u) := \int_0^t e^{(t-\tau)\Delta} u \, d\tau, \tag{1.6}$$

are the bilinear and linear terms, respectively. Our ill-posedness result shows that the third condition (3) of continuity is violated by carefully constructing a particular class of arbitrarily small initial data that produce arbitrarily large solutions in arbitrarily short time. In doing so, we demonstrate that the culprit responsible for generating norm inflation lies in the bilinear term in equation (1.4). Therefore, it is the density u in the Keller–Segel model which exhibits norm inflation. Roughly speaking, the key steps to showing this norm inflation property is to first decompose the integral system, especially the bilinear term, into several parts: one part stemming from the bilinear term responsible for norm inflation and the remaining terms which can be controlled. The a priori estimates for solutions of the Cauchy problem in $\dot{F}_2^{-1,r=2}(\mathbb{R}^2) \times BMO(\mathbb{R}^2)$ is an important ingredient in this step since they are exploited in order to control some of those remaining terms in the decomposition. The $\dot{F}_2^{-1,r>2}(\mathbb{R}^2)$–norm of the solution u in arbitrary short time can then be bounded from below by the norm inflation term and the controlled terms. Thus, this proves the solution map for u is discontinuous at the initial time. We mention that the a priori estimates established here shows the continuity of the bilinear and linear operators (1.6) and such bounds are crucial in proving well-posedness results for the associated Cauchy problem. Naturally, this leads one to seek well-posedness results for (1.1)–(1.3) with initial data in $\dot{F}_2^{-1,2}(\mathbb{R}^2) \times BMO(\mathbb{R}^2)$, and this can certainly be addressed more thoroughly in future investigations.

This manuscript is organized as follows. Section 2 recalls several preliminary definitions, results, and tools from harmonic analysis employed throughout this paper. Then, the statement of our main result is provided at the end of the section. Section 3 establishes the a priori estimates on the bilinear and linear terms stemming from (1.1)–(1.3) with initial data $(u_0, v_0) \in \dot{F}_2^{-1,2}(\mathbb{R}^2) \times BMO(\mathbb{R}^2)$.

Recall that such estimates in these function spaces will play a key role in establishing the ill-posedness of the Keller–Segel model. Section 4 proves the main ill-posedness result by first outlining the main steps in the proof. Nevertheless, for completeness sake and for the reader's convenience, the intermediate steps and preliminary estimates which complement the main steps are then given in the form of several lemmas.

2. Preliminaries and the Main Result

The proof of the main result presented in this paper requires a dyadic Littlewood-Paley decomposition. Let us briefly explain how it can be developed in $\mathbb{R}^2$, but the reader is referred to [1] for further details. Let $\mathcal{S}(\mathbb{R}^2)$ be the Schwartz class and $\varphi(\xi) = \varphi(|\xi|)$ be a smooth function valued in $[0,1]$ such that

$$\text{supp } \varphi \subset \{\xi \in \mathbb{R}^2; \ 3/4 \leq |\xi| \leq 8/3\} \ \text{ and } \ \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \ \xi \neq 0. \tag{2.1}$$

For $f \in \mathcal{S}'(\mathbb{R}^2)$, the space of tempered distributions, we define the homogeneous dyadic block and partial summation operator as follows:

$$\Delta_j f(x) := \mathcal{F}_\xi^{-1}(\varphi(2^{-j}\xi)\widehat{f}(\xi))(x) \ \text{ and } \ S_j f(x) := \sum_{i \leq j-1} \Delta_i f(x) \text{ for all } j \in \mathbb{Z}.$$

Moreover, the Littlewood-Paley decomposition satisfies the following quasi-orthogonal properties:

$$\Delta_i \Delta_j f \equiv 0 \text{ if } |i-j| \geq 2, \quad \Delta_j(S_{i-1}f\Delta_i g) \equiv 0 \text{ if } |i-j| \geq 5. \tag{2.2}$$

Using Bony's decomposition, we can split the product of two functions f and g,

$$fg = T_f g + T_g f + R(f,g), \tag{2.3}$$

where $T_f g = \sum_j S_{j-1}f\Delta_j g$, $T_g f = \sum_j S_{j-1}g\Delta_j f$ and $R(f,g) = \sum_j \sum_{l=-1}^1 \Delta_j f\Delta_{j+l}g$. Particularly, $R(f,g)$ is the *remainder*, and $T_f g$ and $T_g f$ are the *paraproducts*.

Let us define the Triebel–Lizorkin spaces and the related Besov spaces. The $BMO(\mathbb{R}^2)$ space, which is equivalent to $\dot{F}_\infty^{0,2}(\mathbb{R}^2)$, plays an important role in this paper, so we provide an equivalent definition for this space as well.

Definition 2.1. *For $(s,q,r) \in \mathbb{R} \times (1,\infty) \times [1,\infty]$, we define $\dot{B}_q^{s,r}(\mathbb{R}^2)$ to be the set of tempered distributions f such that*

$$\|f\|_{\dot{B}_q^{s,r}(\mathbb{R}^2)} = \|\{2^{js}\|\Delta_j f\|_{L^q(\mathbb{R}^2)}\}_{j \in \mathbb{Z}}\|_{\ell^r} < \infty, \tag{2.4}$$

and we define $\dot{F}_q^{s,r}(\mathbb{R}^2)$ to be the set of tempered distributions f such that

$$\|f\|_{\dot{F}_q^{s,r}(\mathbb{R}^2)} = \|\|\{2^{js}\Delta_j f\}_{j \in \mathbb{Z}}\|_{\ell^r}\|_{L^q(\mathbb{R}^2)} < \infty. \tag{2.5}$$

Furthermore, we define $BMO(\mathbb{R}^2)$ to be the space of tempered distributions f such that

$$\|f\|_{BMO(\mathbb{R}^2)} = \sup_{x \in \mathbb{R}^2, r>0} \left(\frac{1}{r^2} \int_{|y-x|<r} \int_0^{r^2} |\nabla e^{t\Delta}f(y)|^2 dtdy\right)^{\frac{1}{2}} < \infty.$$

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Now we are ready to state our main result.

Theorem 2.2. *For any $2 < r \leq \infty$ and $\delta > 0$, there exists a solution (u, v) to system (1.1)–(1.3) with initial data $(u_0, v_0) \in \dot{F}_2^{-1,r}(\mathbb{R}^2) \times BMO(\mathbb{R}^2)$ satisfying*

$$\|u_0\|_{\dot{F}_2^{-1,r}(\mathbb{R}^2)} \lesssim \delta, \quad \|v_0\|_{BMO(\mathbb{R}^2)} \lesssim \delta,$$

such that for some $0 < T < \delta$, $\|u(T)\|_{\dot{F}_2^{-1,r}(\mathbb{R}^2)} \gtrsim \frac{1}{\delta}$, but $\|v(T)\|_{BMO(\mathbb{R}^2)} \lesssim \delta$.

Remark 2.3. *From this point on, we shall use C and c to denote universal constants which may change from line to line. Both $\mathcal{F}f$ and $\hat{f}$ denote the Fourier transform of f with respect to the spatial variable, while $\mathcal{F}^{-1}$ denotes the inverse Fourier transform. We denote $A \leq CB$ by $A \lesssim B$ and $A \lesssim B \lesssim A$ by $A \sim B$.*

3. A Priori Estimates

In this section, we first give several preliminary lemmas and obtain the bilinear and linear estimates. In order to prove the bilinear and linear estimates below, we require the following result on Carleson measures, cf. [21, Proposition 10.1].

Lemma 3.1. *Let $\{\beta_j(x)\}_j$ be a sequence of measurable functions on $\mathbb{R}^2$ defining a Carleson measure on $\mathbb{Z} \times \mathbb{R}^2$,*

$$\sup_{x_0 \in \mathbb{R}^2, r > 0} \frac{1}{r^2} \sum_{2^j r > 1} \int_{|x - x_0| < r} |\beta_j(x)|^2 dx < \infty. \tag{3.1}$$

Let $h(x) \in L^1$ so that $(1 + |x|)^3 h(x) \in L^\infty$ and $h_j(x) = 2^{2j} h(2^j x)$. Then for any $f \in L^2$, there holds

$$\int_{\mathbb{R}^2} \sum_{j \in \mathbb{Z}} |f * h_j|^2 |\beta_j|^2 dx \leq C \|f\|_{L^2}^2 \sup_{x_0 \in \mathbb{R}^2, r > 0} \frac{1}{r^2} \sum_{2^j r > 1} \int_{|x - x_0| < r} |\beta_j(x)|^2 dx, \tag{3.2}$$

where C does not depend on f, h or $\{\beta_j\}_{j \in \mathbb{Z}}$.

Remark 3.2. *From (2.1), if we denote the kernel of $\psi(\nabla)$ by $h(x)$, then $h(\cdot) \in L^1(\mathbb{R}^2)$. The kernel of $\psi(2^{-j}\nabla)$ is $h_j(x) = 2^{2j} h(2^j x)$ and $(1 + |\cdot|)^3 h(\cdot) \in L^\infty(\mathbb{R}^2)$; else if we let $\varphi(\xi) = \sum_{j \leq 0} \psi(2^{-j}\xi)$, then φ is compactly supported in $\{\xi \in \mathbb{R}^2; |\xi| \leq \frac{8}{3}\}$. If we denote $\mathcal{F}^{-1}(\varphi(\xi))$ by $h(x)$ and $2^{2j} h(2^j x)$ by $h_j(x)$, then it is easy to check that $h(\cdot) \in L^1(\mathbb{R}^2)$ and $(1 + |\cdot|)^3 h(\cdot) \in L^\infty(\mathbb{R}^2)$. Moreover, since each $BMO(\mathbb{R}^2)$ function can be defined equivalently by a Carleson measure and each Carleson measure defines a $BMO(\mathbb{R}^2)$ function, we have that for any $BMO(\mathbb{R}^2)$ function b, $\{\Delta_j b\}_{j \in \mathbb{Z}}$ satisfies the above assumptions for $\{\beta_j\}_{j \in \mathbb{Z}}$. Consequently, we find that for any $BMO(\mathbb{R}^2)$ function g and $L^2(\mathbb{R}^2)$ function f,*

$$\|T_f g\|_{L^2(\mathbb{R}^2)} + \|R(f, g)\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{L^2(\mathbb{R}^2)} \|g\|_{BMO(\mathbb{R}^2)}. \tag{3.3}$$

The definitions and remark above can be found in several books of harmonic analysis, including those on pseudodifferential operators, see for instance [12, Chapter 8, Paraproducts] and other relevant texts such as [7, 23].

The final lemma of this section establishes the bilinear and linear estimates.

Lemma 3.3. *Let $B(u,v)$ and $L(u)$ be defined as in (1.6). Then we have the following estimates:*

(i) $\|B(u,v)\|_{L^2(0,T;L^2(\mathbb{R}^2))} \lesssim \|u\|_{L^2(0,T;L^2(\mathbb{R}^2))}(\|v\|_{L^\infty(0,T;BMO(\mathbb{R}^2))} + \|\nabla v\|_{L^2(0,T;BMO(\mathbb{R}^2))})$,

(ii) $\|B(u,v)\|_{L^\infty(0,T;\dot{F}_2^{-1,2}(\mathbb{R}^2))} \lesssim \|u\|_{L^2(0,T;L^2(\mathbb{R}^2))}(\|v\|_{L^\infty(0,T;BMO(\mathbb{R}^2))} + \|\nabla v\|_{L^2(0,T;BMO(\mathbb{R}^2))})$,

(iii) $\|L(u)\|_{L^\infty(0,T;BMO(\mathbb{R}^2))} + \|\nabla L(u)\|_{L^2(0,T;BMO(\mathbb{R}^2))} \lesssim \|u\|_{L^2(0,T;L^2(\mathbb{R}^2))}$.

Proof. First recall the definition of the bilinear operator $B(u,v)$ and linear operator $L(u)$,

$$B(u,v) = \int_0^t e^{(t-\tau)\Delta} \nabla \cdot (u\nabla v)\, d\tau, \qquad L(u) = \int_0^t e^{(t-\tau)\Delta} u\, d\tau.$$

Since the inner function $u\nabla v$ in the bilinear operator is in product form, we can use Bony's decomposition to express it as the sum of three parts,

$$u\nabla v = T_u \nabla v + R(u, \nabla v) + T_{\nabla v} u. \tag{3.4}$$

Since Riesz transforms are bounded in $L^2(\mathbb{R}^2)$, the energy method, (3.3) and Hölder's inequality imply

$$\left\| \int_0^t e^{(t-\tau)\Delta} \nabla \cdot (T_u \nabla v + R(u, \nabla v))\, d\tau \right\|_{L^2(0,T;L^2(\mathbb{R}^2))} \lesssim \|T_u \nabla v + R(u, \nabla v)\|_{L^1(0,T;L^2(\mathbb{R}^2))}$$
$$\lesssim \|u\|_{L^2(0,T;L^2(\mathbb{R}^2))} \|\nabla v\|_{L^2(0,T;BMO(\mathbb{R}^2))}. \tag{3.5}$$

It remains to bound the last part, $\int_0^t e^{(t-\tau)\Delta} \nabla \cdot (T_{\nabla v} u)\, d\tau$. By the maximal regularity for the heat kernel (cf. [21, Theorem 7.3, p.64]) and because $\dot{F}_2^{-1,2}(\mathbb{R}^2) = \dot{H}^{-1}(\mathbb{R}^2)$, we see that

$$\left\| \int_0^t e^{(t-\tau)\Delta} \nabla \cdot (T_{\nabla v} u)\, d\tau \right\|_{L^2(0,T;L^2(\mathbb{R}^2))} \lesssim \|T_{\nabla v} u\|_{L^2(0,T;\dot{F}_2^{-1,2}(\mathbb{R}^2))}. \tag{3.6}$$

The Minkowski inequality, (2.3), Hölder's inequality, and Young's inequality imply

$$\|T_{\nabla v} u\|_{\dot{F}_2^{-1,2}(\mathbb{R}^2)} \lesssim \Big(\sum_{j\in\mathbb{Z}} 2^{-2j} \|S_j \nabla v \Delta_j u\|_{L^2(\mathbb{R}^2)}^2\Big)^{\frac{1}{2}} \lesssim \Big(\sum_{j\in\mathbb{Z}} \Big(\sum_{i<j-1} 2^{i-j} \|\Delta_i v\|_{L^\infty(\mathbb{R}^2)} \|\Delta_j u\|_{L^2(\mathbb{R}^2)}\Big)^2\Big)^{\frac{1}{2}}$$
$$\lesssim \sup_{i\in\mathbb{Z}} \|\Delta_i v\|_{L^\infty(\mathbb{R}^2)} \Big(\sum_j \|\Delta_j u\|_{L^2(\mathbb{R}^2)}^2\Big)^{\frac{1}{2}} \lesssim \|v\|_{BMO(\mathbb{R}^2)} \|u\|_{L^2(\mathbb{R}^2)}, \tag{3.7}$$

where in the last inequality, we used the fact that $BMO(\mathbb{R}^2) \subset \dot{B}_\infty^{0,\infty}(\mathbb{R}^2)$ and $\dot{B}_2^{0,2}(\mathbb{R}^2) = L^2(\mathbb{R}^2)$. Applying (3.7) to (3.6) and using Hölder's inequality, we obtain

$$\left\| \int_0^t e^{(t-\tau)\Delta} \nabla \cdot (T_{\nabla v} u)\, d\tau \right\|_{L^2(0,T;L^2(\mathbb{R}^2))} \lesssim \|v\|_{L^\infty(0,T;BMO(\mathbb{R}^2))} \|u\|_{L^2(0,T;L^2(\mathbb{R}^2))}.$$

Estimate (ii) is handled similarly i.e.

$$\left\| \int_0^t e^{(t-\tau)\Delta} \nabla \cdot (u\nabla v)\, d\tau \right\|_{L^\infty(0,T;\dot{F}_2^{-1,2}(\mathbb{R}^2))}$$
$$\lesssim \left\| T_u \nabla v + R(u, \nabla v) \right\|_{L^1(0,T;L^2(\mathbb{R}^2))} + \left\| T_{\nabla v} u \right\|_{L^2(0,T;\dot{F}_2^{-1,2}(\mathbb{R}^2))}$$
$$\lesssim \|u\|_{L^2(0,T;L^2(\mathbb{R}^2))} \|\nabla v\|_{L^2(0,T;BMO(\mathbb{R}^2))} + \|u\|_{L^2(0,T;L^2(\mathbb{R}^2))} \|v\|_{L^\infty(0,T;BMO(\mathbb{R}^2))}. \tag{3.8}$$

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By combining the above arguments, we arrive at the estimates (i) and (ii). To prove estimate (iii), we use the embedding $\dot{F}_2^{1,2}(\mathbb{R}^2) \subset BMO(\mathbb{R}^2)$, Plancherel's identity, and Young's inequality to obtain

$$\|L(u)\|_{L^\infty(0,T;BMO(\mathbb{R}^2))} \lesssim \|L(u)\|_{L^\infty(0,T;\dot{F}_2^{1,2}(\mathbb{R}^2))} \lesssim \|u\|_{L^2(0,T;L^2(\mathbb{R}^2))}.$$

Furthermore, by the maximal regularity for the heat kernel and since $\dot{F}_2^{1,2}(\mathbb{R}^2) \subset BMO(\mathbb{R}^2)$, we see that

$$\|\nabla L(u)\|_{L^2(0,T;BMO(\mathbb{R}^2))} \lesssim \|\Delta L(u)\|_{L^2(0,T;L^2(\mathbb{R}^2))} \lesssim \|u\|_{L^2(0,T;L^2(\mathbb{R}^2))}.$$

This completes the proof. $\qquad\square$

4. Proof of Ill-posedness

In this section, for the sake of simplicity, it suffices to show the ill-posedness of the Keller–Segel model in $\dot{F}_2^{-1,r}(\mathbb{T}^2)$ $(r > 2)$ since the ill-posedness result for the non-periodic case can be treated using the methods for maximal functions introduced in [10, 11] to the cutoff function $u_0(x)\phi(x)$. Here, u_0 is given by (4.5) below and ϕ satisfies $\operatorname{supp}\hat{\phi} \subset \{\xi \in \mathbb{R}^2; \ |\xi| \le 1/4\}$, $0 \le \hat{\phi} \le 1$ and $\int_{\mathbb{R}^2} \hat{\phi}\, d\xi \sim 1$.

4.1. Rewriting the Keller-Segel Model

Adopting the ideas from [5], we rewrite the two-dimensional Keller–Segel model by decomposing it into its first approximation, second approximation and remainder terms as follows,

$$u = u_1 - u_2 + y, \qquad v = v_1 + v_2 + z, \tag{4.1}$$

where
$$u_1 := e^{t\Delta}u_0, \quad u_2 := B(u_1, v_1), \quad v_1 := e^{t\Delta}v_0, \quad v_2 := L(u_1). \tag{4.2}$$

Moreover, the remainder terms satisfy the integral equations,

$$y = V_2 + V_1 + V_0, \qquad z = L(y) - L(u_2), \tag{4.3}$$

on $(0, \infty)$ with the initial conditions $(y(0), z(0)) = (0, 0)$,

$$V_2 = -B(y, z), \quad V_1 = B(u_2 - u_1, z) - B(y, v_1 + v_2), \quad V_0 = B(u_2, v_1 + v_2) - B(u_1, v_2). \tag{4.4}$$

4.2. Construction of Initial Data for the Keller–Segel Model

For a fixed small number $\delta > 0$ we define the initial data as follows:

$$\begin{cases} u_0(x) = \dfrac{Q}{\sqrt{\rho}} \displaystyle\sum_{s=1}^{\rho} k_s \cos(k_s x_2), \\[4mm] v_0(x) = \dfrac{1}{\sqrt{\rho Q}} \displaystyle\sum_{s=1}^{\rho} \Big(\cos((1 - k_s)x_2) \Big), \end{cases} \tag{4.5}$$

where the parameters satisfy:

- $k_0 = 2^{M_0}; \quad k_s = 2^s k_0 k_{s-1} = 2^{\frac{(s+1)(s+2M_0)}{2}}$ with $s = 1, 2, \cdots$ and $M_0 \gg 4$.

According to the constructions of u_0 and v_0, it is easy to check that

$$e^{t\Delta}u_0 = \frac{Q}{\sqrt{\rho}} \sum_{s=1}^{\rho} k_s e^{-tk_s^2} \cos(k_s x_2), \quad e^{t\Delta}v_0 = \frac{1}{\sqrt{Q\rho}} \sum_{s=1}^{\rho} e^{-t(1-k_s)^2} \cos((1-k_s)x_2). \tag{4.6}$$

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4.3. Outline of the Proof

In order to effectively communicate the main ideas in the proof of Theorem 2.2, this subsection outlines the main steps. Further details on the intermediate steps are given in the subsections which follow hereafter.

Step 1: Fix a real number $\delta > 0$. Split u_2 from (4.1) into three parts, $u_2 = u_{2,0} + u_{2,1} + u_{2,2}$, where the first term $u_{2,0}$ exhibits norm inflation while $u_{2,1}$ and $u_{2,2}$ are controllable terms.

Step 2: With our careful choice of initial conditions and making appropriate choices for Q, ρ, k_0, and T, we establish the following estimates:

- $\|u_{2,0}(T)\|_{L^2(\mathbb{T}^2)} \gtrsim Q^{\frac{1}{2}}$,

- $\|u_{2,1}(T) + u_{2,2}(T)\|_{\dot{F}_2^{-1,2}(\mathbb{T}^2)} \lesssim Q^{\frac{1}{2}}\rho^{-1}$,

- $\|u_1(T)\|_{\dot{F}_2^{-1,r}(\mathbb{T}^2)} \lesssim \rho^{\frac{1}{r}-\frac{1}{2}}Q$,

- $\|y(T)\|_{\dot{F}_2^{-1,2}(\mathbb{T}^2)} \lesssim \left(T^{\frac{1}{2}} + \rho^{-1}\right) + Q^{Q^3+3}\left(k_0^{-1} + \rho^{-\frac{1}{2}}\right)$.

Step 3: (Norm Inflation) The estimates in Step 2 imply

$$\|u(T)\|_{\dot{F}_2^{-1,r}(\mathbb{T}^2)} \geq \|u_{2,0}(T)\|_{\dot{F}_2^{-1,r}(\mathbb{T}^2)} - \|u_1(T)\|_{\dot{F}_2^{-1,r}(\mathbb{T}^2)} - \|u_{2,1}(T) + u_{2,2}(T) - y(T)\|_{\dot{F}_2^{-1,r}(\mathbb{T}^2)}$$

$$\gtrsim \|u_{2,0}(T)\|_{L^2(\mathbb{T}^2)} - \|u_1(T)\|_{\dot{F}_2^{-1,2}(\mathbb{T}^2)} - \|u_{2,1}(T) + u_{2,2}(T) - y(T)\|_{\dot{F}_2^{-1,2}(\mathbb{T}^2)}$$

$$\gtrsim \|u_{2,0}(T)\|_{L^2(\mathbb{T}^2)} - \rho^{\frac{1}{r}-\frac{1}{2}}Q - Q^{\frac{1}{2}}\rho^{-1} - \left(T^{\frac{1}{2}} + \rho^{-1}\right) - Q^{Q^3+3}\left(k_0^{-1} + \rho^{-\frac{1}{2}}\right)$$

$$\gtrsim Q^{\frac{1}{2}}\left(1 - \rho^{-1} - \rho^{\frac{1}{r}-\frac{1}{2}}Q^{\frac{1}{2}} - Q^{-\frac{1}{2}}(T^{\frac{1}{2}} + \rho^{-1}) + Q^{Q^3+\frac{5}{2}}(k_0^{-1} + \rho^{-\frac{1}{2}})\right)$$

$$\gtrsim Q^{\frac{1}{2}} \gtrsim 1/\delta, \tag{4.7}$$

provided that $\rho^{\frac{1}{r}-\frac{1}{2}}Q^{\frac{1}{2}}$, $Q^{Q^3+5/2}(k_0^{-1} + \rho^{-\frac{1}{2}}) \ll 1$. Hence, for sufficiently large ρ and k_0 and $T \ll Q^{-\frac{1}{2}} \ll \delta$, (4.7) holds thereby showing u exhibits norm inflation. This will complete the proof of the ill-posedness result. Therefore, it remains to establish the estimates listed in step 2. These estimates are provided in the subsequent lemmas below.

4.4. Estimates for u_0, $e^{t\Delta}u_0$, v_0 and $e^{t\Delta}v_0$

Lemma 4.1. *For any $2 \leq r \leq \infty$, we have*

$$\|u_0\|_{\dot{F}_2^{-1,r}(\mathbb{T}^2)} + \|e^{t\Delta}u_0\|_{\dot{F}_2^{-1,r}(\mathbb{T}^2)} \lesssim \rho^{\frac{1}{r}-\frac{1}{2}}Q, \tag{4.8}$$

$$\|v_0\|_{BMO(\mathbb{T}^2)} + \|e^{t\Delta}v_0\|_{BMO(\mathbb{T}^2)} \lesssim Q^{-\frac{1}{2}}. \tag{4.9}$$

Proof. From (4.5), (4.6), $\max\{e^{-tk_s^2}, e^{-t(1-k_s)^2}\} \leq 1$ and Definition 2.1, it suffices to estimate u_0 and v_0.

Estimates for u_0: We prove this for the two endpoints $r = 2$ and $r = \infty$ then obtain the estimate for all intermediate values of r by interpolation. For $r = 2$, by orthogonality and the fact that $\dot{F}_2^{-1,2}(\mathbb{T}^2) = \dot{H}^{-1}(\mathbb{T}^2)$, we obtain

$$\|u_0\|_{\dot{F}_2^{-1,2}(\mathbb{T}^2)} \sim \frac{Q}{\sqrt{\rho}}\left\|\left(\sum_{s=1}^{\rho}\left(\cos(k_sx_2)\right)^2\right)^{\frac{1}{2}}\right\|_{L^2(\mathbb{T}^2)} \lesssim Q. \tag{4.10}$$

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For $r = \infty$, Definition 2.1 implies

$$\|u_0\|_{\dot{F}_2^{-1,\infty}(\mathbb{T}^2)} \sim \frac{Q}{\sqrt{\rho}} \Big\| \sup_{s \in \{1,\cdots,\rho\}} |\cos(k_s x_2)| \Big\|_{L^2(\mathbb{T}^2)} \lesssim \frac{Q}{\sqrt{\rho}}, \tag{4.11}$$

where in each dyadic annulus $\{\xi \in \mathbb{R}^2; \frac{3}{4}2^j \leq |\xi| \leq \frac{8}{3}2^j\}$, there exists at most one k_s ($s = 1,\cdots,\rho$). Combining (4.10) and (4.11) and by interpolation, we have that

$$\|u_0\|_{\dot{F}_2^{-1,r}(\mathbb{T}^2)} \leq \|u_0\|_{[\dot{F}_2^{-1,2}(\mathbb{T}^2),\dot{F}_2^{-1,\infty}(\mathbb{T}^2)]_{\frac{2}{r}}} \lesssim \rho^{\frac{1}{r}-\frac{1}{2}} Q \quad \text{for } 2 \leq r \leq \infty. \tag{4.12}$$

Estimates for v_0: From the construction of v_0 and the definition of BMO, we have

$$\|v_0\|_{BMO(\mathbb{T}^2)} \lesssim \|e^{t\Delta} \nabla v_0\|_{L^2(0,\infty;L^\infty(\mathbb{T}^2))}$$

$$\lesssim \frac{1}{\sqrt{\rho Q}} \Big\| \sum_{s=1}^{\rho} e^{-t(1-k_s)^2}(1-k_s) \sin\Big((1-k_s)x_2\Big) \Big\|_{L^2(0,\infty;L^\infty(\mathbb{T}^2))}$$

$$\lesssim \frac{1}{\sqrt{\rho Q}} \Big(\int_0^\infty \sum_{s=1}^{\rho} \sum_{\ell=1}^{\rho} e^{-ct(k_s^2+k_\ell^2)} k_s k_\ell \, dt \Big)^{\frac{1}{2}}$$

$$\lesssim \frac{1}{\sqrt{\rho Q}} \Big(\sum_{j=1}^{\rho} \int_0^\infty e^{-ctk_j^2} k_j^2 \, dt \Big)^{\frac{1}{2}} \lesssim Q^{-\frac{1}{2}}, \tag{4.13}$$

since $\displaystyle \sum_{j=1}^{\rho} \sum_{1 \leq i < j} e^{-ct(k_i^2+k_j^2)} k_i k_j \lesssim \sum_{j=1}^{\rho} e^{-ctk_j^2} k_j^2$. $\qquad\qquad\square$

Lemma 4.2. *For any $T > 0$, we have*

$$\|e^{t\Delta} u_0\|_{L^2(0,T;L^2(\mathbb{T}^2))} \lesssim Q, \tag{4.14}$$

$$\|e^{t\Delta} v_0\|_{L^\infty(0,T;BMO(\mathbb{T}^2))} \lesssim Q^{-\frac{1}{2}}, \tag{4.15}$$

$$\|e^{t\Delta} \nabla v_0\|_{L^2(0,T;BMO(\mathbb{T}^2))} \lesssim Q^{-\frac{1}{2}}. \tag{4.16}$$

Proof. To prove estimate (4.14), we have

$$\|e^{t\Delta} u_0\|_{L^2(0,T;L^2(\mathbb{T}^2))} = \frac{Q}{\sqrt{\rho}} \Big\| \sum_{s=1}^{\rho} e^{-tk_s^2} k_s \cos(k_s x_2) \Big\|_{L^2(0,T;L^2(\mathbb{T}^2))}$$

$$\lesssim \frac{Q}{\sqrt{\rho}} \Big\| \sum_{s=1}^{\rho} e^{-tk_s^2} k_s \Big\|_{L^2(0,T)} \lesssim \frac{Q}{\sqrt{\rho}} \Big(\int_0^T \sum_{s=1}^{\rho} \sum_{\ell=1}^{\rho} e^{-ct(k_s^2+k_\ell^2)} k_s k_\ell \, dt \Big)^{\frac{1}{2}}$$

$$\lesssim Q. \tag{4.17}$$

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Next, we prove (4.15). By recalling the definition of the $BMO(\mathbb{T}^2)$ space, we see that for any $t > 0$,

$$\|e^{t\Delta}v_0\|_{BMO(\mathbb{T}^2)} = \left(\sup_{x_0 \in \mathbb{T}^2, r > 0} \frac{1}{r^2} \int_0^{r^2} \int_{|x - x_0| < r} \left| e^{\sigma\Delta}(\nabla e^{t\Delta}v_0) \right|^2 d\sigma dx \right)^{\frac{1}{2}}$$

$$\lesssim \frac{1}{\sqrt{\rho Q}} \left(\int_0^\infty \left\| \sum_{s=1}^{\rho} e^{-(\sigma+t)(1-k_s)^2}(1-k_s)\sin(x_2 - k_s x_2) \right\|_{L^\infty(\mathbb{T}^2)}^2 d\sigma \right)^{\frac{1}{2}}$$

$$\lesssim \frac{1}{\sqrt{\rho Q}} \left(\int_0^\infty \left(\sum_{s=1}^{\rho} e^{-c(\sigma+t)k_s^2} k_s \right)^2 d\sigma \right)^{\frac{1}{2}} \lesssim Q^{-\frac{1}{2}}. \tag{4.18}$$

Hence,

$$\|e^{t\Delta}v_0\|_{L^\infty(0,T;BMO(\mathbb{T}^2))} \lesssim Q^{-\frac{1}{2}}. \tag{4.19}$$

At last, we prove the third estimate which follows similarly to the proof of (4.18) since

$$\|e^{t\Delta}\nabla v_0\|_{L^2(0,T;BMO(\mathbb{T}^2))} \lesssim \|e^{t\Delta}\nabla v_0\|_{L^2(0,T;L^\infty(\mathbb{T}^2))} \lesssim Q^{-\frac{1}{2}}. \tag{4.20}$$

This completes the proof of the lemma. $\qquad\square$

4.5. Estimates for $u_{2,0}$, $u_{2,1}$, $u_{2,2}$ and v_2

From (4.2) and the construction of the initial data, we can rewrite the approximation terms as follows: $u_2 = u_{2,0} + u_{2,1} + u_{2,2}$, with

$$u_{2,0} = \frac{Q^{\frac{1}{2}}}{2\rho} \sum_{s=1}^{\rho} \int_0^t e^{-(t-\tau)} e^{-\tau(k_s^2 + (1-k_s)^2)} k_s(k_s - 1) \cos x_2 \, d\tau, \tag{4.21}$$

$$u_{2,1} = \frac{Q^{\frac{1}{2}}}{2\rho} \sum_{s=1}^{\rho} \sum_{\ell \neq s} \int_0^t \frac{k_s(k_\ell - 1)(1 + k_s - k_\ell)}{e^{(t-\tau)(1 + k_s - k_\ell)^2 + \tau(k_s^2 + (1-k_\ell)^2)}} \cos((1 + k_s - k_\ell)x_2) \, d\tau, \tag{4.22}$$

$$u_{2,2} = \frac{Q^{\frac{1}{2}}}{2\rho} \sum_{s=1}^{\rho} \sum_{\ell=1}^{\rho} \int_0^t \frac{k_s(k_\ell - 1)(1 - k_s - k_\ell)}{e^{(t-\tau)(1 - k_s - k_\ell)^2 + \tau(k_s^2 + (1-k_\ell)^2)}} \cos((1 - k_s - k_\ell)x_2) \, d\tau. \tag{4.23}$$

The following estimates are concerned with the norm inflation terms.

Lemma 4.3. *Suppose that* $k_1^{-2} = 2^{-2-4M_0} \ll T \ll 1$, *then we have*

$$\|u_{2,0}(T)\|_{\dot{F}_2^{-1,r}(\mathbb{T}^2)} \sim Q^{\frac{1}{2}}, \tag{4.24}$$

$$\|u_{2,0}\|_{L^2(0,T;L^2(\mathbb{T}^2))} \lesssim T^{\frac{1}{2}} Q^{\frac{1}{2}}. \tag{4.25}$$

Proof. By the definition of $u_{2,0}$, $k_1^{-2} \ll T \ll 1$ for $t = T$, we have that

$$u_{2,0}(T) = \frac{Q^{\frac{1}{2}}}{2\rho} \sum_{s=1}^{\rho} e^{-T}(1 - e^{T(1 - k_s^2 - (1-k_s)^2)}) \frac{k_s(k_s - 1)}{k_s^2 + (1-k_s)^2} \cos x_2 \sim Q^{\frac{1}{2}} \cos x_2. \tag{4.26}$$

By noticing that the frequency of $\cos x_2$, i.e. $(0, 1)$, is localized in $\{\frac{3}{4}2^{-1} \leq |\xi| \leq \frac{8}{3}2^{-1}\} \cup \{\frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$, (4.24) and (4.25) follow from (4.26) by direct computations. $\qquad\square$

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Lemma 4.4. *With the definition of the initial data, the term $u_{2,1}$ satisfies the bounds*

$$\|u_{2,1}\|_{L^2(0,T;L^2(\mathbb{T}^2))} + \|u_{2,1}\|_{L^\infty(0,T;\dot{F}_2^{-1,2}(\mathbb{T}^2))} \lesssim \frac{Q^{\frac{1}{2}}}{\rho}, \tag{4.27}$$

$$\|u_{2,2}\|_{L^2(0,T;L^2(\mathbb{T}^2))} + \|u_{2,2}\|_{L^\infty(0,T;\dot{F}_2^{-1,2}(\mathbb{T}^2))} \lesssim \frac{Q^{\frac{1}{2}}}{\rho}. \tag{4.28}$$

Proof. It suffices to prove (4.27). Note that for any $s \neq \ell$, $|1 + k_s - k_\ell| \sim k_s + k_\ell$. Thus,

$$\|u_{2,1}\|_{L^2(0,T;L^2(\mathbb{T}^2))} \lesssim \frac{Q^{\frac{1}{2}}}{\rho} \Big\| \sum_{s=1}^{\rho} \sum_{\ell \neq s} \int_0^t e^{-c(t-\tau)(k_s+k_\ell)^2 - c\tau(k_s^2+k_\ell^2)} (k_s+k_\ell) k_s k_l \, d\tau \Big\|_{L^2(0,T)}$$

$$\lesssim \frac{Q^{\frac{1}{2}}}{\rho} \Big\| \sum_{s=1}^{\rho} \sum_{l \neq s} \int_0^t e^{-ct(k_s+k_\ell)^2} (k_s+k_\ell) k_s k_\ell \, d\tau \Big\|_{L^2(0,T)}$$

$$\lesssim \frac{Q^{\frac{1}{2}}}{\rho} \Big\| \sum_{s=1}^{\rho} e^{-ctk_s^2} t k_s^2 k_{s-1} + \sum_{\ell=1}^{\rho} e^{-ctk_\ell^2} t k_\ell^2 k_{\ell-1} \Big\|_{L^2(0,T)}$$

$$\lesssim \frac{Q^{\frac{1}{2}}}{\rho} \Big(\sum_{s=1}^{\rho} \frac{k_{s-1}}{k_s} + \sum_{\ell=1}^{\rho} \frac{k_{l-1}}{k_\ell} \Big) \lesssim \frac{Q^{\frac{1}{2}}}{\rho}. \tag{4.29}$$

Moreover,

$$\|u_{2,1}\|_{L^\infty(0,T;\dot{F}_2^{-1,2}(\mathbb{T}^2))} \lesssim \frac{Q^{\frac{1}{2}}}{\rho} \Big\| \sum_{s=1}^{\rho} \sum_{\ell \neq s} \int_0^t e^{-c(t-\tau)(k_s+k_\ell)^2 - c\tau(k_s^2+k_\ell^2)} k_s k_l \, d\tau \Big\|_{L^\infty(0,T)}$$

$$\lesssim \frac{Q^{\frac{1}{2}}}{\rho} \Big\| \sum_{s=1}^{\rho} e^{-ctk_s^2} t k_s k_{s-1} + \sum_{\ell=1}^{\rho} e^{-ctk_\ell^2} t k_\ell k_{\ell-1} \Big\|_{L^\infty(0,T)}$$

$$\lesssim \frac{Q^{\frac{1}{2}}}{\rho} \Big(\sum_{s=1}^{\rho} \frac{k_{s-1}}{k_s} + \sum_{\ell=1}^{\rho} \frac{k_{l-1}}{k_\ell} \Big) \lesssim \frac{Q^{\frac{1}{2}}}{\rho}. \tag{4.30}$$

Estimate (4.27) follows from (4.29) and (4.30), and (4.28) follows from similar arguments. Thus, this completes the proof of the lemma. $\qquad\square$

Lemma 4.5. *The linear term v_2 satisfies the estimate,*

$$\|v_2\|_{L^\infty(0,T;BMO(\mathbb{T}^2))} + \|\nabla v_2\|_{L^2(0,T,BMO(\mathbb{T}^2))} \lesssim \frac{T^{\frac{1}{2}}Q}{\sqrt{\rho}}. \tag{4.31}$$

Proof. From (4.2) and construction of the initial data, we get

$$\begin{cases} v_2 = L(u_1) = \int_0^t e^{(t-\tau)\Delta + \tau\Delta} u_0 \, d\tau = \frac{Q}{\sqrt{\rho}} \sum_{s=1}^{\rho} e^{-tk_s^2} t k_s \cos(k_s x_2), \\ \\ \nabla v_2 = L(\nabla u_1) = -\frac{Q}{\sqrt{\rho}} \sum_{s=1}^{\rho} e^{-tk_s^2} (0, t k_s^2)^T \sin(k_s x_2). \end{cases} \tag{4.32}$$

Recalling that $L^\infty(\mathbb{T}^2) \subset BMO(\mathbb{T}^2)$, we have

$$
\begin{cases}
\|v_2\|_{L^\infty(0,T;BMO(\mathbb{T}^2))} \lesssim \|v_2\|_{L^\infty(0,T;L^\infty(\mathbb{T}^2))} \lesssim \dfrac{T^{\frac{1}{2}}Q}{\sqrt{\rho}} \sum_{s=1}^{\rho} e^{-tk_s^2} t^{\frac{1}{2}} k_s \lesssim \dfrac{T^{\frac{1}{2}}Q}{\sqrt{\rho}}, \\[2ex]
\|\nabla v_2\|_{L^2(0,T;BMO(\mathbb{T}^2))} \lesssim \|\nabla v_2\|_{L^2(0,T;L^\infty(\mathbb{T}^2))} \lesssim \dfrac{Q}{\sqrt{\rho}} \| \sum_{s=1}^{\rho} e^{-tk_s^2} t k_s^2 \|_{L^2(0,T)} \lesssim \dfrac{T^{\frac{1}{2}}Q}{\sqrt{\rho}}.
\end{cases}
$$

Therefore, we finish the proof. $\qquad\square$

4.6. Estimates on the Remainder Terms y and z

As described earlier, we need to estimate the remainder terms especially the term y and show they remain relatively small, but as illustrated below, this requires a more delicate estimate on the term u_1 than what was achieved in the earlier subsection. In order to do so, we must control the terms in smaller time scales then sum their contributions to obtain the desired estimate on the global time scale. As before, this technical procedure was was developed in [5]. Let $k_\rho^{-2} = T_0 < T_1 < T_2 < \ldots < T_\beta = k_0^{-2}$ where $\beta = Q^3$, $T_\alpha = k_{\rho_\alpha}^{-2}$, $\rho_\alpha = \rho - \alpha Q^{-3}\rho$, and $\alpha = 0, 1, 2, \ldots, \beta$.

Lemma 4.6. *Let $u_1 = e^{t\Delta}u_0$. Then for any $\alpha \in \{0, 1, \cdots, Q^3\}$, we have*

$$
\|u_1\|_{L^2(T_\alpha, T_{\alpha+1}; L^2(\mathbb{T}^2))} = \left\| e^{t\Delta}u_0 \right\|_{L^2(T_\alpha, T_{\alpha+1}; L^2(\mathbb{T}^2))} \lesssim \frac{Q}{\sqrt{\rho}}\left(1 + \sqrt{\rho}Q^{-\frac{3}{2}}\right). \tag{4.33}
$$

Particularly, from $T_0 = k_\rho^{-2}$, we have

$$
\|u_1\|_{L^2(0,T_0; L^2(\mathbb{T}^2))} \lesssim \frac{Q}{\sqrt{\rho}} \quad and \quad \|v_1\|_{L^2(0,T_0; L^2(\mathbb{T}^2))} \lesssim \frac{1}{\sqrt{Q\rho}}. \tag{4.34}
$$

Proof. It suffices to prove (4.33). By Plancherel's identity and by the construction of the initial datum u_0, we get

$$
\|e^{t\Delta}u_0\|_{L^2(\mathbb{T}^2)} \sim \frac{Q}{\sqrt{\rho}}\Big(\sum_{s=1}^{\rho} k_s^2 e^{-2tk_s^2}\Big)^{\frac{1}{2}} := I. \tag{4.35}
$$

It suffices to estimate

$$
\left(\int_{T_\alpha}^{T_{\alpha+1}} I^2 dt\right)^{\frac{1}{2}} \lesssim \frac{Q}{\sqrt{\rho}}\Big(\big(\sum_{s=1}^{\rho_{\alpha+1}} + \sum_{s=\rho_{\alpha+1}+1}^{\rho_\alpha} + \sum_{s=\rho_\alpha+1}^{\rho}\big)\big(e^{-2T_\alpha k_s^2} - e^{-2T_{\alpha+1}k_s^2}\big)\Big)^{\frac{1}{2}}
$$
$$
\lesssim \frac{Q}{\sqrt{\rho}}\sqrt{1 + \rho Q^{-3} + 1} \lesssim \frac{Q}{\sqrt{\rho}}(1 + \sqrt{\rho Q^{-3}}).
$$

Thus we finish the proof. $\qquad\square$

Recall that the definition of the remainder terms as found in equations (4.3). A key step in our norm inflation argument relies on controlling y and verifying it remains small.

Lemma 4.7. *For $\alpha \in \{0, 1, 2, \cdots, \beta\}$, $T > T_\beta = k_0^{-2}$, $T_\beta < Q^{-2}$ and $\rho \gg Q^6$*

$$
\|y(T)\|_{L^2(0,T;L^2(\mathbb{T}^2))} \lesssim \left(T^{\frac{1}{2}} + \rho^{-1}\right) + Q^{Q^3+2}\left(k_0^{-1} + \rho^{-\frac{1}{2}}\right).
$$

Proof. With the help of the previous lemmas including the continuity of the linear and bilinear operators as given in Lemma 3.3, we establish some important bounds on the terms from (4.3).

$$\|B(u_2, v_1 + v_2)\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))}$$
$$\lesssim \|u_2\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))}\Big(\|v_1 + v_2\|_{L^\infty(0,T_{\alpha+1};BMO(\mathbb{T}^2))} + \|\nabla v_1 + \nabla v_2\|_{L^2(0,T_{\alpha+1};BMO(\mathbb{T}^2))}\Big)$$
$$\lesssim \Bigg(\underbrace{Q^{\frac{1}{2}}\rho^{-1}}_{\text{Lemma 4.4}} + \underbrace{T_{\alpha+1}^{\frac{1}{2}}Q^{\frac{1}{2}}}_{\text{Lemma 4.3}}\Bigg)\Bigg(\underbrace{Q^{-\frac{1}{2}}}_{\text{Lemma 4.2}} + \underbrace{T_{\alpha+1}^{\frac{1}{2}}Q\rho^{-\frac{1}{2}}}_{\text{Lemma 4.5}}\Bigg) \lesssim (\rho^{-1} + T_\beta^{\frac{1}{2}});$$

$$\|B(u_1, v_2)\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))}$$
$$\lesssim \|u_1\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))}\Big(\|v_2\|_{L^\infty(0,T_{\alpha+1};BMO(\mathbb{T}^2))} + \|\nabla v_2\|_{L^2(0,T_{\alpha+1};BMO(\mathbb{T}^2))}\Big)$$
$$\lesssim \underbrace{Q}_{\text{Lemma 4.2}}\ \underbrace{T_{\alpha+1}^{\frac{1}{2}}Q\rho^{-\frac{1}{2}}}_{\text{Lemma 4.5}}$$
$$\lesssim T_\beta^{\frac{1}{2}};$$

$$\|B(u_2 - u_1, z)\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))}$$
$$\lesssim \|u_2 - u_1\|_{L^2(0,T_\alpha;L^2(\mathbb{T}^2))}\Big(\|z\|_{L^\infty(0,T_\alpha;BMO(\mathbb{T}^2))} + \|\nabla z\|_{L^2(0,T_\alpha;BMO(\mathbb{T}^2))}\Big)$$
$$+ \|u_2 - u_1\|_{L^2(T_\alpha,T_{\alpha+1};L^2(\mathbb{T}^2))}\Big(\|z\|_{L^\infty(T_\alpha,T_{\alpha+1};BMO(\mathbb{T}^2))} + \|\nabla z\|_{L^2(T_\alpha,T_{\alpha+1};BMO(\mathbb{T}^2))}\Big)$$
$$\lesssim \Bigg(\underbrace{T_\alpha^{\frac{1}{2}}Q^{\frac{1}{2}} + Q^{\frac{1}{2}}\rho^{-1} + Q}_{\text{Lemmas 4.2,4.3 and 4.4}}\Bigg)\Big(\|z\|_{L^\infty(0,T_\alpha;BMO(\mathbb{T}^2))} + \|\nabla z\|_{L^2(0,T_\alpha;BMO(\mathbb{T}^2))}\Big)$$
$$+ \Bigg(\underbrace{T_{\alpha+1}^{\frac{1}{2}}Q^{\frac{1}{2}} + Q^{\frac{1}{2}}\rho^{-1}}_{\text{Lemmas 4.2 and 4.4,}} + \underbrace{Q^{-\frac{1}{2}}}_{\text{Lemma 4.6}}\Bigg)\Big(\|z\|_{L^\infty(0,T_{\alpha+1};BMO(\mathbb{T}^2))} + \|\nabla z\|_{L^2(0,T_{\alpha+1};BMO(\mathbb{T}^2))}\Big)$$
$$\lesssim Q\Big(\|z\|_{L^\infty(0,T_\alpha;BMO(\mathbb{T}^2))} + \|\nabla z\|_{L^2(0,T_\alpha;BMO(\mathbb{T}^2))}\Big)$$
$$+ Q^{-\frac{1}{2}}\Big(\|z\|_{L^\infty(0,T_{\alpha+1};BMO(\mathbb{T}^2))} + \|\nabla z\|_{L^2(0,T_{\alpha+1};BMO(\mathbb{T}^2))}\Big);$$

$$\|B(y, v_1 + v_2)\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))}$$
$$\lesssim \|y\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))}\Big(\|v_1 + v_2\|_{L^\infty(0,T_{\alpha+1};BMO(\mathbb{T}^2))} + \|\nabla v_1 + \nabla v_2\|_{L^2(0,T_{\alpha+1};BMO(\mathbb{T}^2))}\Big)$$
$$\lesssim \|y\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))}\Bigg(\underbrace{Q^{-\frac{1}{2}}}_{\text{Lemma 4.2}} + \underbrace{T_{\alpha+1}^{\frac{1}{2}}Q\rho^{-\frac{1}{2}}}_{\text{Lemma 4.5}}\Bigg) \lesssim Q^{-\frac{1}{2}}\|y\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))};$$

$$\|B(y, z)\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))}$$
$$\lesssim \|y\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))}\Big(\|z\|_{L^\infty(0,T_{\alpha+1};BMO(\mathbb{T}^2))} + \|\nabla z\|_{L^2(0,T_{\alpha+1};BMO(\mathbb{T}^2))}\Big);$$

$\|z\|_{L^\infty(0,T_{\alpha+1};BMO(\mathbb{T}^2))} + \|\nabla z\|_{L^2(0,T_{\alpha+1};BMO(\mathbb{T}^2))}$

$$\lesssim \|y\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))} + \|u_2\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))} \lesssim \|y\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))} + \underbrace{T_{\alpha+1}^{\frac{1}{2}}Q^{\frac{1}{2}} + Q^{\frac{1}{2}}\rho^{-1}}_{\text{Lemmas 4.3 and 4.4}}$$

$$\lesssim \|y\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))} + Q^{\frac{1}{2}}(T_\beta^{\frac{1}{2}} + \rho^{-1}).$$

For a suitable choice of $c < 1$ and for any $\alpha \in \{0, 1, \cdots, \beta\}$, we set

$$A_{\alpha+1} := \|y\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))} + c\|z\|_{L^\infty(0,T_{\alpha+1};BMO(\mathbb{T}^2))} + c\|\nabla z\|_{L^2(0,T_{\alpha+1};BMO(\mathbb{T}^2))},$$

and combine the above estimates to obtain

$$\begin{aligned}
A_{\alpha+1} &\lesssim Q^{\frac{1}{2}}\left(T_{\alpha+1}^{\frac{1}{2}} + \rho^{-1}\right) + Q\left(\|z\|_{L^\infty(0,T_\alpha;BMO(\mathbb{T}^2))} + \|\nabla z\|_{L^2(0,T_\alpha;BMO(\mathbb{T}^2))}\right) \\
&\quad + Q^{-\frac{1}{2}}\left(\|z\|_{L^\infty(0,T_{\alpha+1};BMO(\mathbb{T}^2))} + \|\nabla z\|_{L^2(0,T_{\alpha+1};BMO(\mathbb{T}^2))}\right) \\
&\quad + (Q^{-\frac{1}{2}} + c)\|y\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))} \\
&\quad + \|y\|_{L^2(0,T_{\alpha+1};L^2(\mathbb{T}^2))}\left(\|z\|_{L^\infty(0,T_{\alpha+1};BMO(\mathbb{T}^2))} + \|\nabla z\|_{L^2(0,T_{\alpha+1};BMO(\mathbb{T}^2))}\right).
\end{aligned} \tag{4.36}$$

This implies that

$$A_{\alpha+1} \lesssim (T_\beta^{\frac{1}{2}} + \rho^{-1}) + QA_\alpha + A_{\alpha+1}^2. \tag{4.37}$$

Therefore, (4.34), (4.37) and an iteration argument imply

$$A_0 \lesssim Q\rho^{-\frac{1}{2}}, \quad A_\beta \lesssim Q^{\beta+1}\left(T_\beta^{\frac{1}{2}} + \rho^{-\frac{1}{2}}\right),$$

and hence,

$$\|y\|_{L^2(0,T_\beta;L^2(\mathbb{T}^2))} + c\|z\|_{L^\infty(0,T_\beta;BMO(\mathbb{T}^2))} + c\|\nabla z\|_{L^2(0,T_\beta;BMO(\mathbb{T}^2))} \lesssim Q^{Q^3+1}\left(k_0^{-1} + \rho^{-\frac{1}{2}}\right). \tag{4.38}$$

If we iterate (4.37) and (4.38), we have that for $T > T_\beta = k_0^{-2}$,

$$\begin{aligned}
&\|y\|_{L^2(0,T;L^2(\mathbb{T}^2))} + c\|z\|_{L^\infty(0,T;BMO(\mathbb{T}^2))} + c\|\nabla z\|_{L^2(0,T;BMO(\mathbb{T}^2))} \\
&\qquad \lesssim Q^{\frac{3}{2}}\left(T^{\frac{1}{2}} + \rho^{-1}\right) + QA_\beta \lesssim \left(T^{\frac{1}{2}} + \rho^{-1}\right) + Q^{Q^3+2}\left(k_0^{-1} + \rho^{-\frac{1}{2}}\right).
\end{aligned} \tag{4.39}$$

This completes the proof of this Lemma. $\qquad\square$

Combining the bilinear estimates for $B(u,v)$, the linear estimates for $L(u)$, and (4.39), we prove that

$$\|y(T)\|_{\dot{F}_2^{-1,2}(\mathbb{T}^2)} \lesssim \left(T^{\frac{1}{2}} + \rho^{-1}\right) + Q^{Q^3+3}\left(k_0^{-1} + \rho^{-\frac{1}{2}}\right). \tag{4.40}$$

This completes the steps required in proving Theorem 2.2.

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Acknowledgements

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